ON THE BEHAVIOR OF SOLUTION OF NONLINEAR EQUATIONS

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Abstract. In this paper we establish of the Wiener criterion for solution the mixed boundary problem for nonlinear elliptic equation of second order.

1. Introduction and preliminaries.

Let us consider the problem

$$A(u) = \frac{d}{dx} a_i(x, u, u_x) + a(x, u, u_x) = 0 \tag{1}$$

$$u|_{\Gamma_1} = 0, \quad a_i(x, u, u_x)\, cos(n, x)|_{\Gamma_2} = 0 \tag{2}$$

in the domain Ω. Let Ω be an open set in R^n with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, and let $\begin{cases} A(u) = 0 \\ u - f \in W^1_{m,0}(\Omega) \end{cases}$ be a fixed number. The Dirichlet conditions are fulfilled in Γ_1, and Neumaun conditions are fulfilled in Γ_2, and $0 \in \overline{\Gamma_1} \cap \Gamma_2$. Moreover we suppose that domain Ω satisfying isoperimetric conditions. Assume that the functions $a_i(x, u, p)$, $a(x, u, p)$ are defined for $x \in \overline{\Omega}$ and arbitrary u,p, are measurable and satisfy the following conditions

$$\begin{array}{l} a_i\ (x, u, p)\, p_i \geq v\, |p|^m - d\, |u|^m - g \\ |a_i\ (x, u, p)| \leq v\, |p|^{m-1} - b\, |u|^{m-1} + l \\ |a_i\ (x, u, p)| \leq\ |p|^{m-1} + d\, |u|^{m-1} + f \\ \sum\limits_{i=1}^{n} [a_i\ (x, u, p) - a_i\ (x, u, q\ )]\ \ (p_i - q_i) \geq c\, |p - q|^m \end{array} \tag{3}$$

Here $v, d, g, b, l, c, f-$ const's.

The function $u(x) \in W^1_{m,0}(\Omega)$ is said to be a generalized solution of problem (1),(2) if it satisfies the following integral identity

$$\int\limits_{\Omega} [a_i(x, u, u_x)\, \varphi_{x_i} + a(x, u, u_x)\, \varphi]\, dx = 0 \tag{4}$$

for $\forall\varphi \in W^1_{m,0}(\Omega)$. Here $W^1_{m,0}(\Omega)$ is a closure in $W^1_m(\Omega)$ of functions from $C_0^\infty(\partial\Omega\backslash\Gamma_2)$.

The principal model operator is the p-laplacian

$$-\Delta_m u = -div\ \left(|\nabla u|^{m-2}\, \nabla u\right)$$

A boundary point x_0 of bounded Ω is regular if the solution $u$to the mixed boundary problem

$$\begin{cases} A(u) = 0 \\ u - f \in W^1_{m,0}(\Omega) \end{cases}$$

has the limit value $f(x_0)$ at whenever $f \in W^1_{m,0}(\Omega)$ is continuous in the closure of Ω.

In [1] Wiener proved that in the case of the Laplacian the regularity of a boundary point $x_0 \in \partial\Omega$ can be characterized by a so called Wiener test. In [2] Littman, Stampacchia and Weinberger showed that the same Wiener test identifies the regular boundary points whenever A is a uniformly elliptic linear operator with bounded measurable coefficients.

For general nonlinear operators the classical Wiener test has to be modified so that the type m of the operator A is involved. In [3] Maz'ya established that the boundary point x_0 is regular if $W_m(R^n \backslash \Omega, x_0) = +\infty$, where $W_m(R^n \backslash \Omega, x_0)$ is a Wiener type integral. Later in [4] Gariepy and Ziemer extended this result to a very general class of equation.

In [5] Skrypnik established necessary condition of regularity of a boundary points for general class of equations. However this is necessary condition coincidenced with a sufficient conditions only in case $m = 2$.

The question whether regular boundary point of Ω can be characterized by using the Wiener test has been a well known open problem in nonlinear potential theory [6]. In case the Dirichlet condition the problem was partly solved in the affirmative when [7] proved that if m equals n. At last in [8] the established the necessity part of the Wiener test for all $m \in (1, n]$ in case the Dirichlet condition.

In case mixed boundary condition we in [9] established a sufficient and a necessary condition of regularity of the boundary points to a very general class of equations. However this is necessary condition coincidenced with a sufficient conditions only in case $m = 2,\ m = n$, or $m > n - 1$. Unfortunately, their method cannot be extended to cover all values $1 < m \leq n$.

In this paper we establish the necessity part of the Wiener test for all $m \in (1, n]$ and prove:

Theorem1.1. ***Let*** Ω ***satisfy isoperimetric conditions.*** A ***finite boundary point*** $x_0 \in \overline{\Gamma_1} \cap \Gamma_2$ ***is regular if and only if***

$$W_m\left(B_1\left(x\right) \backslash \Omega, x_0\right) = \int\limits_0^{1/2} \left[C_m\left(\Gamma_1, B_i\left(x_0\right)/\Omega, \Gamma_2\right) t^{m-n}\right]^{\frac{1}{m-1}} \frac{dt}{t} = \infty.$$

An immediate corollary is:

Corollary1.1. ***The regularity depends only on*** n ***and***m***, not on the operator*** A ***itself.***

Note that no boundedness assumption on Ω was made in the theorem above, for we extend the definition of regularity for boundary points of unbounded sets below. Also observe that the similar question could be asked also for $m > n$. However, then all points are regular and the corresponding Wiener integral always diverges because singletons are of positive m-conductuvity.

The uniformly elliptic linear equations are included in our presentation. Let us also point out that this methods can be applied to the equations with weights so that the results of this paper are easily generalized to cover the equations considered in [10]. .

Let us give definition of m-conductivity. Denote by F bounded subsets of open set Ω closed in Ω, and by G bounded open subsets of Ω.

The set $K = G/F$ is called a conductor. By $V_\Omega\left(K\right)$ we will denote the class of functions $\{f \in C^\infty\left(\Omega\right),\ f\left(x\right) = 1$, when $x \in F$, and $f\left(x\right) = 0$ when $x \in \Omega/G\}$.

The following quantity will be called a m -conductivity of the conductor K

$$C_m(K) \equiv C_m(F,\Omega,G) = inf\left\{ \int |\nabla f|^m dx \ : \quad f \in V_\Omega(K) \right\}$$

Let us formulate conditions for domain. Let $v_{M,m}(t)$ be the greatest lower bound of $C_m(K)$ in the set of all the conductors $k = G/F$, satisfying the condition $m_n(F) \geq t, m_n(G) \leq M$, where m_n- Lebesque measure. Consider the domains Ω, for which the following condition is fulfilled

$$\lim_{t\to+0} t^{-\alpha m} v_{M,m}(t) > 0, \text{ where } \quad \alpha \geq \frac{n-m}{nm} \tag{5}$$

In case of $m=1$ this condition coincides with classical isoperimetric conditions. Therefore we condition (5) will be call isoperimetric condition.

There is another variant of the Wiener criterion problem , known among specialists in nonlinear potential theory. A set $\Omega \subset R^n$is said to be m-thin at a point if $x_0 \in R^n$ if $W_m(B_t(x_0)\backslash\Omega, x_0) < +\infty$. This concept of thinness was first considered in nonlinear potential theory by [11]. Note that because each sigleton is of m-conductuvity zero it does not have any effect on the $(\bar{B} = \bar{B}(x_0,r))$ m-thinness of Ω whether or not the point x_0 is in Ω. Also it is trivial Ω that is m -thin at each point in the complement of $\overline{\Omega}$. The sets that are m-thin at x_0 were characterze as those sets whose complements are A-fine neighborhoods of x_0. Here A-fine refers to the fine topology of A-superharmonic functions. However it remained unsolved of the m -thinness is equivalent to the so called Cartan property: " there is an A-superharmonic function u in neighborhood of x_0 such that $\lim_{\substack{x\to x_0\\ x\in\Omega}} inf u(x) > u(x_0)$.

Theorem 1.2. ***Let*** $\Omega \subset R^n$ ***and***$x_0 \in \overline{\Omega}/\Omega$***. Then*** A ***is*** m***-thin at*** x_0 ***if and only if there is an of***A ***-superharmonic function*** u ***in a neighborhood of***$x_0 \in \overline{\Gamma_1} \cap \Gamma_2$ ***such that***

$$\lim_{\substack{x\to x_0\\ x\in\Omega}} inf u(x) > u(x_0) \tag{6}$$

The proofs of **Theorems 1.1** and **1.2** are based on pointwise estimates of solutions to

$$Au = \mu \tag{7}$$

with a Radon measure μ on the right side.

The letter c stands for various constants. For an open (closed) ball $B = B(x_0,r)$ $\left(\overline{\mathrm{B}} = \overline{B}(x_0,r)\right)$ with radius r an center x_0 and $\sigma > 0$ we write σB for the open ball with radius σr. The barred integral $sign \fint_E f dx$ stands for the integral average $|E|^{-1} \int_E f dx$, where $|E|$ is Lebesgue measure of E.

The operator T is defined such that for each $\varphi \in C_0^\infty(\partial\Omega\backslash\Gamma_2)$

$$Tu(\varphi) = \int_\Omega Au\nabla\varphi dx,$$

where $u \in W^1_{m,0,loc}(\Omega)$. In other words

$$Tu = -divAu$$

in the sence of distributions.

A solution $u \in W^1_{m,0,loc}(\Omega)$ to the equation

$$Tu = 0 \tag{8}$$

always has a continuous representative; we call continuous solutions $u \in W^1_{m,0,loc}(\Omega) \cap C(\Omega)$ of (7) A-harmonic in Ω.

A lower semicontinuous function $u : \Omega \to (-\infty, \infty]$ is A-superharmonic if u is not identically infinite in each component of Ω, and if for all open $D \subset\subset \Omega$and $h \in C\left(\overline{D}\right)$, A-harmonic in D, $h \le u$ on ∂D implies $h \le u$ in D. A function v is A-subharmonic if -v is A-superhamonic.

Clearly, $min\,(u, v)$and $\lambda u + \sigma$ are$A$-superharmonic if u and v are, and. The following proposition connects A-superharmonic functions with supersolutions of (7).

Proposition 1. ***(i) If** $u \in W^1_{m,0,loc}(\Omega)$ **is such that** $Tu \ge 0$**, then there is an** A**-superharmonic function** v **such that** $u = v$ **a.e. Moreover,***

$$v(x) = ess \lim_{y \to x} inf\, v(y) \quad \text{for all } x \in \Omega \tag{9}$$

(ii) If** v **is** A**-superharmonic, then (9) holds. Moreover,** $Tv \ge 0$ **if** $v \in W^1_{m,0,loc}(\Omega)$ **.

(iii) If**v **is** A**-superharmonic and locally bounded, then**$v \in W^1_{m,0,loc}(\Omega)$ **and** $Tv \ge 0$**.

The prove this proposition analogously the prove proposution 2.7 in [10] .

Let $u \in W^1_{m,0,loc}(\Omega)$ be an A -superharmonic function in Ω. Then it follows from Proposition 1 that $\mu = Tu$ is a nonnegative Radon measure on Ω. If $\Omega'$is an open subset of Ω with $u \in W^1_m(\Omega')$, the restriction v of μ to Ω' belongs to the dual space $\left(W^1_{m,0}(\Omega')\right)'$ of $W^1_{m,0}(\Omega)$. By a standard approximaton we see that

$$\int_{\Omega} Au \nabla\varphi\, dx = \int_{\Omega'} \varphi\, d\mu \tag{10}$$

for any test function $\varphi \in W^1_{m,0}(\Omega')$, where the last integral is the duality pairing between $\varphi \in W^1_{m,0}(\Omega')$ and $v \in \left(W^1_{m,0}(\Omega')\right)'$.

For the reader's convenience we record here an appropriate form of weak Harnack inequality (see [9],[12] and Proposition 1 above).

Lemma 1.1. Let $B = B(x_0, z)$ and let u be a nonnegative A- superharmonic function in$3B$. If $q > 0$ is such that $q(n-p) > n(p-1)$, then

$$\left(\fint_{2B} u^q dx \right)^{\frac{1}{q}} \le c \, \inf_B u$$

where $c = c(n, m, q) > 0$.

Later we establish estimates for A- superharmonic solutions of (7) in terms of the Wolff potential

$$W^{\mu}_{1,m}(x_0, r) = \int_o^r \left(\frac{\mu(B(x_0, t))}{t^{n-m}} \right)^{1/m-1} \frac{dt}{t}$$

One easily infers hat $W^{\mu}_{1,2}(x_0, \infty)$ is the Newtonian potential of μ. This estimation gives a solid link between the two nonlinear potential theories.

Theorem1.3. ***Suppose that*** u ***is a nonnegative*** A***- superharmonic function in***$B\left(x_0,3r\right)$***. If***$\mu = Tu$***, then***

$$c_1 W^{\mu}_{1,m}\left(x_0,r\right) \leq u\left(x_0\right) \leq c_2 inf u + c_3 W^{\mu}_{1,m}\left(x_0,2r\right)$$

where c_1, c_2, c_3 ***are positive constants, depending only on*** $n, m,$ ***and the structural constants. In particular,*** $u\left(x_0\right) < \infty$ ***if and only if*** $W^{\mu}_{1,m}\left(x_0,r\right) < \infty$***.***

Generally speaking is possible indicate that the necessity of the Wiener test follows from an estimate like that in Theorem1.3. In the present paper we choose another route, more natural and direct.

Moreover, we deduce from Theorem1.3 a Harnack inequality for positive solutions to (7), where the measure μ satisfies for some positive constants and ε

$$\mu\left(B\left(x,r\right)\right) \leq c r^{n-m+\varepsilon} \tag{11}$$

whenever $B\left(x,r\right)$ is a ball. Iterating the Harnack inequality in a standard way one sees that the solutions are Holder continuous ; moreover, we show that if the solutions of $Tu = \mu$ is Holder continuous, then μ satisfies a restriction like (10). As a further consequence of Theorem1.3 we characterise continuous A- superharmonic functions in terms of the corresponding Wolff potentials.

2. A -POTENSIALS AND m-CONDUCTUVITY ESTIMATES

If $r > 0$ and $r \leq R$, then there is a positive constant c_i depending only on n and m such that for all $x \in R^n$

$$c^{-1} r^{n-m} \leq C_m\left(B\left(x,r\right), B\left(x,R\right), B\left(x,r\right)\right) \leq c r^{n-m}$$

We say that a conductor K is of m-conductuvity zero if

$$C_m\left(F \cap B, 2B, G \cap B\right) = 0$$

whenever B is an open ball inR^n. Equivalently K is of m-conductuvity zero if and only if

$$C_m\left(F \cap \Omega, \Omega, G \cap \Omega\right) = 0$$

for all open sets Ω. Moreover, for $m < n$ this is further equivalent to

$$C_m\left(F, R, G\right) = 0.$$

We say that a property holds m-quasieverywhere in Ω if it holds in Ω except on a set of m-conductuvity zero. It is well known that each function $u \in W^1_{m,0}\left(\Omega\right)$ has a representative for which the limit $\lim\limits_{r\to 0} \fint_{B(x,r)} u dy$ exists and equals $u(x)$ m -quasieverywhere in Ω.These representative are called m-refined. In what follows we usually consider only the m-refined representatives of functions in $W^1_{m,0}\left(\Omega\right)$. Note that for a locally bounded A-superharmonic function u the limit above exists and is equal to $u\left(x\right)$ for every x.

Suppose that F, G be a subset of Ω. For $x \in \Omega$ let

$$R^1_{F,G}\left(\Omega, A\right)\left(x\right) = inf u\left(x\right)$$

where the infimum is taken over all nonnegative A-superharmonic functions u in Ω such that $u \geq 1$ on $F u = 0$ on $\Omega \backslash G$. The lower semicontinuous regularization

$$R^1_{F,G}(\Omega,A)(x) = \underset{r\to 0}{lim}\ \underset{B_r}{inf}\ R^1_{F,G}(\Omega,A)$$

of $R^1_{F,G}(\Omega,A)$ is called the A-potential of F in Ω. The A-potential $\overline{R}^1_{F,G}(\Omega,\mathrm{A})$ is A-superharmonic in Ω and A-harmonic in $\Omega\backslash\bar{F}$. If Ω is a bounded and $F,G,\subset\subset\Omega$, then the A-potential u of F belongs to$W^1_{m,0}(\Omega)$ and

$$C_m(F,\Omega,G) \le \int\limits_{\Omega} |\nabla u|^m dx \le k_1^m C_m(F,\Omega,G),$$

(see [13]).

Now we derive estimates for A-superharmonic functions in terms of their Wolff potentials. Because an A-superharmonic function does not necessarily belong to $W^1_{m,0,loc}(\Omega)$, we extend the definition for the operator T. If u is an A-superharmonic function in Ω. Then we define

$$Tu(\varphi) = \int\limits_{\Omega} \underset{k\to\infty}{lim} A(min(u,k))\nabla\varphi\ dx$$

$\varphi \in W^1_{m,0}(\Omega)$. By [14] $\underset{k\to\infty}{lim} A(min(u,k))$is locally integrable and hence $-Tu$ is its divergence. Since $min(u,k) \in W^1_{m,0,loc}(\Omega)$ and $min(u,k) = min(u,j)$ a.e. in $\{u < min(k,j)\}$, the limit exists. It is equal to $A(u)$ if $u \in W^1_{1,0,loc}$, which is always the case if $m > 2-1\backslash n$.

If u is A-superharmonic in Ω, there is nonnegative Radon measure μ such that in Ω, and conversely, given a finite measure μ in bounded Ω, there is A-superharmonic function u such that $Tu = \mu$ in Ω and $min(u,k) \in W^1_{m,0}(\Omega)$ for all integers k.

We proof auxiliary estimate.

Lemma2.1. Suppose that u is A-superharmonic in a ball $B_{2r}(x)$ and $\mu = Tu$. If a is real constant,$d > 0$ and $m-1 < \gamma < n(m-1)/(n-m+1)$, then there are constants $q = q(m,\gamma)$and

$c > 0$ such that

$$\left(d^{-r}r^{-n}\int\limits_{B_r\cap(u>a)}(u-a)^r dx\right)^{m/q} \le cd^{-r}r^{-n}\int\limits_{B_{2r}\cap(u>a)}(u-a)^r dx + cd^{1-m}r^{m-n}\mu(B_{2r}) \quad ,$$

provided that

$$|B_{2r}\cap\{u>a\}| < \frac{1}{2}d^{-r}\int\limits_{B_r\cap(u>a)}(u-a)^r dx \tag{12}$$

<u>**Proof.**</u> We assume that u is locally bounded and hence $u \in W^1_{m,0,loc}(B_{2r})$, without loss of

generality that $a = 0$. Let $q = \frac{m\gamma}{m-\gamma/(m-1)}$. Notice that $m < q < \frac{mn}{n-m} = m*$.

Using (10) we obtain

$$d^{-r}\int\limits_{B_r\cap(0<u<d)} u^r dx \le |B_r\cap\{u>0\}| \le |B_{2r}\cap\{u>0\}| \le \frac{1}{2}d^{-r}\int\limits_{B_r\cap(u>0)} u^r dx$$

therefore

$$d^{-r}\int\limits_{B_r\cap(u>0)} u^r dx \le 2d^{-r}\int\limits_{B_r\cap(u>d)} u^r dx \le c\int\limits_{B_r}\omega^q dx \quad , \tag{13}$$

where $\omega = \left(1+d^{-1}u^+\right)^{r/q}-1$. Note that $\nabla\omega = \frac{\gamma}{qd}\left(1+d^{-1}u+\right)^{r/q-1}\nabla u^+$.

Let a cut off function $\eta\in C_0^\infty\left(B_{2r}\right)$ such that $0\le\eta\le 1,\quad \eta=1$ on B_r and $|\nabla\eta|\le 2/r$. Using Sobolev inequality we have

$$\left(r^{-n}\int\limits_{B_r}\omega^q dx\right)^{m/q} \le cr^{m-n}\int\limits_{B_{2r}}|\nabla\omega|^m\eta^m dx + cr^{m-n}\int\limits_{B_{2r}}\omega^m|\nabla\eta|^m dx. \tag{14}$$

By substituting the test function $\varphi = \left(1-\left(1+d^{-1}u^+\right)^{1-\tau}\right)u\eta^m$, where $\tau=\gamma/\left(m-1\right)$, the continuation our estimate, using Young's the quality and (12) we obtain

$$r^m\int\limits_{B_{2r}}\omega^m|\nabla\eta|^m dx \le c_3 d^{-r}\int\limits_{B_{2r}\cap(u>0)} u^r dx \quad . \tag{15}$$

Now we remove the assumption that u is locally bounded. For $k>d$ we write $u_k = min\left(u,k\right)$ and $\mu_k = Tu_k$. Then (12) holds for u_k if k is large enough. Hence by the estimates (13)-(15) we arrive at the estimate

$$\left(d^{-\gamma}r^{-n}\int\limits_{B_r\cap\{u>0\}} u_k^\gamma dx\right)^{m/q} \le c_4 d^{-\gamma}r^{-n}\int\limits_{B_{2r}\cap\{u>0\}} u_k^\gamma dx + c_4 d^{1-m}r^{m-n}\mu_k\ \left(supp\eta\right),$$

where $c_4>0$. Now letting $k\to\infty$ and using the weak convergence of μ_k to μ we conclude the proof.

Theorem2.1. ***Suppose that*** u ***is a nonnegative***A ***-superhamonic function in***$B_{2r}\left(x_0\right)$***. If***$\mu = Tu$***, then for all*** $\gamma>m-1$ ***we have that***

$$u\left(x_0\right)\le c\left(\fint\limits_{B_r(x_0)} u^\gamma dx\right)^{1/\gamma} + cW_{1,m}^\mu\left(x_0,2r\right),$$

where $c>0$ ***depends at structure.***

Proof. Let $\gamma>n\left(m-1\right)/(n-m+1)$, fix a constant $\delta\in(0,1)$ to be a specified later, $B_l = B_{r_l}\left(x_0\right)$, where $r_j = 2^{1-j}r$. We define a sequence a_j. Let $a_0=0$ and for $j\ge 0$

$$a_{j+1} = a_l + \delta^{-1}\left(r_l^{-n}\int\limits_{B_{i+1}\cap\{u>a_j\}}\left(u-a_j\right)^\gamma dx\right)^{1/\gamma}$$

Using Lemma2.1 and accompany estimates we obtain

$$a_k - a_1 \le a_{k+1}-a_1 = \sum_{j=1}^k\left(a_{j+1}-a_1\right)\le\frac{1}{2}a_k + c\sum_{j=1}^k\left(\frac{\mu\left(B_j\right)}{r_j^{n-m}}\right)^{1/(m-1)}$$

and hence

$$\underset{k\to\infty}{lim}\, a_k \le 2a_1 + c\sum_{j=1}^{\infty}\left(\frac{\mu\left(B_j\right)}{r_j^{n-m}}\right)^{1/(m-1)} \le c\left(\fint_{B_1} u^{\gamma}dx\right)^{1/\gamma} + cW_{1,m}^{\mu}\left(x_0, 2r\right)$$

Now the theorem follows by $inf\, u \le a_j \quad for\ j = 1, 2, ...$ and for$u$ is lower semicontinuous we conclude that $u\left(x_0\right) \le \underset{j\to\infty}{lim}\, \underset{B_j}{inf}\, u \le \underset{J\to\infty}{lim}\, inf\, a_J$.

Proof of Theorem1.3. The first inequality establishe analogously [10]. The second inequality follows from Theorem2.1 because by the weak Harnack inequality in Lemma1.1. We may pick $\gamma\left(n, m\right) > m-1$ such that

$$\left(\fint_{B_r} u^{\gamma}dx\right)^{1/\gamma} \le c\left(\fint_{B_{2r}} u^{\gamma}dx\right)^{1/\gamma} \le c\, \underset{B_r}{inf}\, u.$$

Corollary2.1. ***Let*** u ***be an*** A ***-superharmonic function in*** R^n ***with*** $\underset{R^n}{inf}\, u = 0$***. If*** $\mu = Tu$***, then***

$$c_1 W_{1,m}^{\mu}\left(x_0; \infty\right) \le u\left(x_0\right) \le c_2 W_{1,m}^{\mu}\left(x_0; \infty\right) \quad ,$$

where $_1$ ***and***$_2$ ***are positive constants, depending only on*** n, m ***and the structural constants.***

Proof of the Theorem1.2. The sufficiency part we was establishe in another paper. We are going to prove the necessity. Let $K = G/F$ be m-thin at $x \notin K$. We may assume that K is open. Write, $B_j = B_{2-j}\left(x_0\right),\ \ r_j = 2^{-j}$, and$K_j = K \cap B_j$. Let$\alpha \ge 2$ be an integer, to be specified later. Let $u = R^1_{F_a, G_a}\left(B_{a-2} : A\right)$ be the A-potential of K_a in $B_{\alpha-2}$ and$\mu = Tu$. Then $u \ge 1$ on K_α and it remains to prove that $u\left(x_0\right) < 1$, for some α. Using some estimates $\mu\ \left(B_j\right)$ we obtain from Theorem1.3 that

$$u(x_0) \le c\inf_{B_\alpha} u + cW_{1,m}^{\mu}(x_0, r_{\alpha-1}) \le c\sum_{j=\alpha-1}^{\infty}\left(\frac{C_m\left(F_j, B_{j-1}, G_j\right)}{r_j^{n-m}}\right)^{1/(m-1)} \le \frac{1}{2}$$

Using Theorem1.2 we have that the Cartan property characterizes fine topologies in nonlinear potential theory. Recall that the A-fine topology is the coarsest topology inR^n that makes all A-superharmonic functions in R^ncontinuous.

Theoem2.2. ***Suppose that*** $\Omega \subset R^n$ ***and*** $x_0 \in \overline{\Omega}$***. Then the following are equivalent: 1)***x_0 ***is not an*** A***-fine limit point of; 2)*** Ω ***is p-thin*** x_0***; 3)(Cartan property) There is an*** A***-superharmonic function*** u ***in a neighborhood of*** x_0 ***such that; 4) There are open neighborhood*** U ***and*** V ***of*** x_0 ***such that*** $R^1_{F\cap U, G\cap U}(V, A) < 1$***.***

Proof this Theorem follows from Theorem1.2.

Next we are ready to prove Theorem1.1. The notice that the define boundary regularity we give in [14].

Proof of Theorem1.1. Suppose that. If x_0 is an isolated boundary point, it never is regular as easily follows by using the maximum principle and the removability of singleton for bounded A-harmonic functions. Hence we are free to assume that x_0 is an accumulation point of. Because E is m-thin at x_0, we now infer from Theorem1.2. that there are balls, such that and an$A$-superharmonic function u in B_2 such that, in and. Next, choose a function such that in and that $\varphi = 1$ in a neighborhood of x_0. Consider the upper Perron

solution taken in the open set. Because the set of the irregular boundary points is of conductuvity zero and because it follows from the generalized comparison principle that in. In particular,.

Hence x_0 is not regular boundary point of. Since that barrier characterization for regularity implies that the regularity is a local property, it follows that is not a regular boundary point of. Theorem1.1 is proved.

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